doi: 10.1111/j.1467-9469.2006.00547.x
 Vol 34: 615–625, 2007

Perfect Simulation for Length-interacting Polygonal Markov Fields in the Plane

M. N. M. VAN LIESHOUT

Centrum voor Wiskunde en Informatica

T. SCHREIBER

Faculty of Mathematics and Computer Science, Nicolaus Copernicus University

ABSTRACT. The purpose of this paper was to construct perfect samplers for length-interacting Arak–Clifford–Surgailis polygonal Markov fields in the plane with nodes of order 2 (*V*-shaped nodes). This is achieved by providing for the polygonal fields a hard core marked point process representation with individual points carrying polygonal loops as their marks, so that the coupling from the past and clan of ancestors routines can be adopted.

Key words: Arak process, clan of ancestors algorithm, coupling from the past, length-interacting polygonal Markov field, marked point process, perfect simulation

1. Introduction

Following the ground breaking paper by Propp & Wilson (1996), which presented exact samplers for a range of discrete probability distributions including the Ising model at critical temperature, a variety of exact or perfect simulation methods has been developed for absolutely continuous probability distributions. In this paper, we are particularly interested in such methods for polygonal Markov fields observed in planar windows (Arak & Surgailis, 1989).

Our interest in developing these algorithms is motivated by the fact that the polygonal Markov fields seem to constitute a natural and promising prior for image segmentation purposes. This was first noted by Clifford & Middleton (1989), whereas the first sampler was developed by Clifford & Nicholls (1994). Even though their original sampler was rather slow, it has been recently re-worked and applied to real data by Paskin & Thrun (2005). A completely different algorithm based on the notion of disagreement loops (Schreiber, 2005) has recently been developed by Kluszczyński *et al.* (2006) and was successfully applied to image segmentation (Kluszczyński *et al.*, 2005).

All the abovementioned samplers were, however, based on the classical Metropolis–Hastings and Gibbs-sampling schemes, and the research was mainly concentrated on elaborating new efficient moves whereas the corresponding rates of convergence remain unknown. In this context, it is important to develop tools allowing for perfect simulation of the polygonal Markov fields. The present paper is the first step in this direction. It deals only with the low-temperature regime for length-interacting fields, that is, for a sufficiently large pre-factor β for the length element in the Hamiltonian in (2) below, yet we plan to extend the applicability of the perfect scheme to the high-temperature and area-interacting regimes as well.

The idea underlying the perfect sampling algorithm presented in this article is to represent the polygonal field as a marked point process, with individual points carrying polygonal loops as their marks. The interaction in this representation turns out to be repulsive and based on a hard core inter-contours exclusion rule.

This paper is outlined as follows. In section 2.1, we review the construction of the length-interacting Arak process with empty boundary condition, then present a survey of marked point and object processes in section 2.2. In section 3, we re-formulate the length-interacting Arak process as a hard object process, and derive the mark distribution. The perfect samplers are discussed in the subsequent section 4. We summarize our results in section 5.

2. Polygonal Markov fields and Markov object processes

A realization of a planar polygonal Markov field in a bounded observation window consists of a finite number of disjoint polygons. The shape and number of the polygons are random. Such configurations also arise as realizations of object processes, the objects being polygons. As the polygons are assumed to be disjoint, we can consider them as hard, that is non-intersecting, objects. It is often convenient to rephrase an object process as a marked point process, by associating with each object a typical point and considering its (shifted) shape as the mark.

It is the purpose of this section to present a brief survey of polygonal Markov fields and object processes. In the next section, we shall rephrase a particular polygonal Markov field, the *length-interacting Arak process with V-shaped nodes* that we are primarily interested in, as a marked point process. Typical realizations of this model are given in Figs 1 and 2.

2.1. Length-interacting polygonal Markov fields

The first example of a polygonal Markov field was provided by Arak (1982), further developments are due to Arak & Surgailis (1989), Arak & Surgailis (1991) and Arak *et al.* (1993). In this paper, we restrict our attention to fields with V-shaped nodes, as in the basic Arak model (Arak, 1982).

The Arak–Clifford–Surgailis polygonal fields enjoy a number of striking mathematical properties which make them particularly suitable not only for statistical applications but also for theoretical analysis both from the viewpoint of stochastic geometry (Schreiber, 2005) and statistical mechanics (Nicholls, 2001; Schreiber, 2006), where additional motivation is due to strong analogies between polygonal Markov fields with V-shaped nodes and the two-dimensional Ising model. An interesting feature of these polygonal fields is the two-dimensional germ Markov property that the conditional behaviour of the field inside a

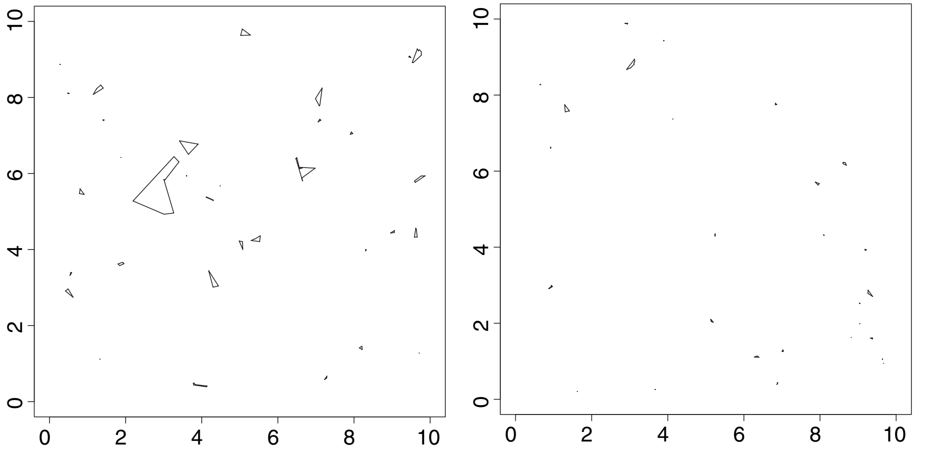

Fig. 1. Perfect samples from the length-interacting Arak process (2) in $D=[0,10]^2$ by algorithm 2 for $\beta=2$ (left) and $\beta=3$ (right).

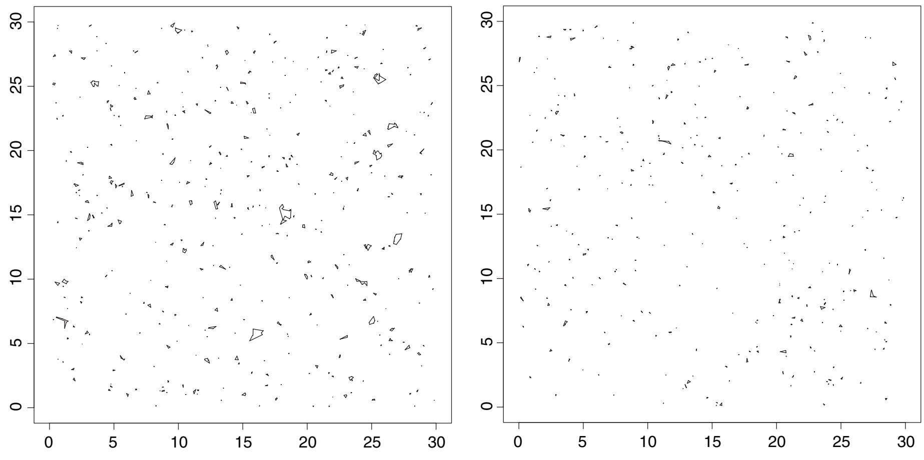

Fig. 2. Perfect samples from the length-interacting Arak process (2) in $D=[0,30]^2$ by algorithm 1 for $\beta=2$ (left) and $\beta=3$ (right).

piecewise smooth closed curve depends on the outside configuration only through the trace of this configuration on the boundary (intersection points and intersection directions). Moreover, the Arak, and indeed a much wider class of polygonal Markov fields, are *consistent* in the sense that the model constructed in an open, bounded convex subset of another such set coincides with the corresponding restriction of the model constructed in the superset (see Arak & Surgailis, 1989). In fact, a lot more is known in this setting, in particular the one-dimensional linear sections of the field turn out to coincide with the corresponding sections for appropriate Poisson line processes, the probability distribution is known in closed form, and a special *dynamic representation* is available for the process in terms of the evolution of one-dimensional particle systems tracing the polygonal boundaries of the field in two-dimensional space-time (see Arak & Surgailis, 1989 for details).

To proceed with a formal description of our setting, let $D\subseteq\mathbb{R}^2$ be a bounded open set of strictly positive Lebesgue measure with piecewise smooth boundary ∂D, to remain fixed throughout this article. Define the family Γ_D of admissible polygonal configurations in D to consist of all planar graphs γ in D such that:

(P1) $\gamma\cap\partial D=\emptyset$;
(P2) all the vertices of γ are of degree 2;
(P3) the edges of γ do not intersect;
(P4) no two edges of γ are co-linear.

In words, γ consists of a finite number of disjoint polygons fully contained in D and possibly nested – see Figs. 1 and 2 for typical realizations of a Γ_D-valued process.

For a finite collection $(l)=\{l_i\}_{i=1}^n$, $n\in\mathbb{N}_0$, of straight lines l_i intersecting D, denote by $\Gamma_D(l)$ the family of $\gamma\in\Gamma_D$ that use (l) as their skeleton in the sense that $\gamma\subseteq\bigcup_{i=1}^n l_i$ and $\gamma\cap l_i$ is a single interval of a strictly positive length for each $l_i, i=1,\dots,n$, possibly with some isolated points added (note that these arise as intersections of edge-extending lines with other edges of the polygonal field).

Let Λ_D be the restriction to D of a Poisson line process Λ with intensity measure given by the standard isometry-invariant Lebesgue measure μ on the space of straight lines in $\mathbb{R}^2$. Then, the basic *polygonal Arak process* $\mathcal{A}_D$ on D is defined by

$$\mathbb{P}\left(\mathcal{A}_D \in G\right) = \frac{\mathbb{E}\sum_{\gamma \in \Gamma_D(\Lambda_D) \cap G} \exp(-2\ell(\gamma))}{\mathbb{E}\sum_{\gamma \in \Gamma_D(\Lambda_D)} \exp(-2\ell(\gamma))} \tag{1}$$

for all $G \subseteq \Gamma_D$ that are Borel measurable with respect to the Hausdorff distance topology, with $\ell(\cdot)$ standing for the Euclidean length measure. Finally, for any $\beta > 0$, the *length-interacting Arak process* $\mathcal{A}_D^{[\beta]}$ in D is determined in distribution by

$$\frac{\mathrm{d}\mathcal{L}(\mathcal{A}_D^{[\beta]})}{\mathrm{d}\mathcal{L}(\mathcal{A}_D)}[\gamma] := \frac{\exp(-\beta\ell(\gamma))}{\mathbb{E}\exp\left(-\beta\ell\left(\mathcal{A}_D\right)\right)}, \tag{2}$$

with $\mathcal{L}(\cdot)$ standing for the law of the argument random object. The reader is referred to (Arak & Surgailis, 1989; Arak *et al.*, 1993) for further details. Models such as (2) whose density, up to a proportionality constant, is expressed in exponential form are known in the language of statistical mechanics as *Gibbsian modifications* of the reference distribution, in this case that of the polygonal Arak process. The negative of the exponent is referred to as the *Hamiltonian* or *energy*, and scalar pre-factors (here β) as *inverse temperature*. Observe that the term *length-interacting*, also borrowed from the language of statistical mechanics, admits a natural interpretation stemming from the fact that large β in (2) tends to favour low-energy configurations leaving little room for entropic fluctuations whereas small β makes the energy factor dominated by the entropic one, in analogy with physical thermodynamic systems.

Note that in the literature on consistent polygonal fields one usually considers free rather than empty boundary conditions as imposed in this paper, that is to say, usually the polygonal contours are allowed to be chopped off by the boundary ∂D whereas here we forbid them to touch the boundary [cf. condition (**P1**)]. We chose to work in this setting as the empty boundary object is better suited for our further purposes. It should be emphasized that the properties mentioned in the beginning of this section for $\mathcal{A}_D$ do not hold for general length-interacting or other Gibbsian modifications of the Arak–Clifford–Surgailis fields.

2.2. *Marked point processes*

Let M be a complete separable metric space and take D as above. A planar *marked point process* $\mathcal{M}_D$ with positions in D and marks in M is a point process on $D \times M$ such that the process of unmarked points is finite (Daley & Vere-Jones, 1988). In other words, realizations of Y are of the form $(y) = \{y_1 = (x_1, m_1), \dots, y_n = (x_n, m_n)\}$, where $n \in \mathbb{N}_0$, $x_i \in D$ and $m_i \in M$ for all $i = 1, \dots, n$, with $x_i \neq x_j$ for $i \neq j$.

Let ν_M be a probability measure on the Borel σ-algebra $\mathcal{B}(M)$. We shall restrict attention to marked point processes that are absolutely continuous with respect to the distribution of a unit rate Poisson process $\mathcal{P}_D$ on D marked independently and identically according to ν_M.

The *Papangelou conditional intensity* of a marked point process $\mathcal{M}$ at $(x, m) \in (D \times M) \setminus (y)$ is defined as

$$\lambda((x, m); \{(x_i, m_i)\}_{i=1}^n) := \frac{(\mathrm{d}\mathcal{L}(\mathcal{M})/\mathrm{d}\mathcal{L}(\mathcal{P}_{\mathcal{D}}))[\{(x_i, m_i)\}_{i=1}^n \cup \{(x, m)\}]}{(\mathrm{d}\mathcal{L}(\mathcal{M})/\mathrm{d}\mathcal{L}(\mathcal{P}_{\mathcal{D}}))[\{(x_i, m_i)\}_{i=1}^n]} \tag{3}$$

whenever $(\mathrm{d}\mathcal{L}(\mathcal{M})/\mathrm{d}\mathcal{L}(\mathcal{P}_{\mathcal{D}}))[\{(x_i, m_i)\}_{i=1}^n] > 0$, and arbitrarily (say 0) otherwise. In other words, (3) may be interpreted heuristically as the conditional probability of finding a point at $\mathrm{d}x$ with mark $\mathrm{d}\nu_M(m)$ conditional on the configuration elsewhere being $\{(x_i, m_i)\}_{i=1}^n$.

Henceforth, we shall assume the following properties to hold:

- **(M1)** $\mathrm{d}\mathcal{L}(\mathcal{M})/\mathrm{d}\mathcal{L}(\mathcal{P}_D)$ is *hereditary*, that is, if marked point pattern (y) is assigned a strictly positive value, so are its subsets;
- **(M2)** *local stability*, that is, the Papangelou conditional intensity is bounded from above by some finite constant $\lambda > 0$.

The mark m of a point at x may be a parametrization of a geometric, planar object $Z(m)$ translated to the location x. Such an object is said to be hard if it cannot overlap other objects. In that case,

$$\lambda((x,m);\{(x_i,m_i)\}_{i=1}^n)=0$$

whenever

$$[x+Z(m)]\cap[\cup_{i=1}^n x_i+Z(m_i)]\neq\emptyset.$$

The generic example is to take a constant conditional intensity whenever there is no overlap, which corresponds to the conditional law

$$\mathcal{L}\left(\mathcal{M}\right)=\mathcal{L}\left(\mathcal{P}_D \mid \forall(x,m),(x',m')\in\mathcal{P}_D : x\neq x' \Rightarrow x+Z(m)\cap x'+Z(m')=\emptyset\right) \tag{4}$$

of a Poisson process given there is no overlap between the objects in it. Clearly, the map $m\rightarrow Z(m)$ has to be such that the conditioning event is measurable. Note that $\mathcal{M}$ is Markov (Ripley & Kelly, 1977) with respect to the overlapping object relation, in the Ripley–Kelly sense that (3) depends only on those (x_i,m_i) for which $x+Z(m)\cap x_i+Z(m_i)\neq\emptyset$. This 'local' property implies a *spatial Markov property*: for any Borel subset A of $D\times M$, the conditional distribution of $\mathcal{M}$ restricted to A given the configuration on the complement of A depends only on $\mathcal{M}$ restricted to the subset of marked points not in A whose associated object overlaps one represented by some marked point in A (Ripley & Kelly, 1977; Van Lieshout, 2000). Note that these notions of Markovianity do not coincide with the germ-Markovianity discussed in section 2.1 above, yet the polygonal Markov fields do enjoy the Markov property in all these senses.

A marked point process is said to be *repulsive* if $\lambda(\cdot;\cdot)$ is decreasing in its second argument with respect to set inclusion, *attractive* if it is increasing. Indeed, in case of repulsion, the more marked points there are, the harder it is to introduce yet another one, and the smaller the conditional intensity. We shall see in the next section that the length-interacting Arak process is repulsive, which is good news for the design of efficient perfect simulation algorithms.

3. Marked point process representation for polygonal Markov fields

In this section, we follow the route outlined in the beginning of section 2 and show that the length-interacting Arak process is a Markov hard object process. To do so we recall that, as argued in section 2.2 in (Schreiber, 2005), the polygonal field $\mathcal{A}_D^{[\beta]}$ admits a so-called *polymer representation* on the space of closed contours in D. Then, we shall represent the object by its leftmost point and a shifted contour, so as to obtain the desired marked point process representation (4).

Let $\mathcal{C}_D$ be the set of all closed polygonal contours in D which do not touch the boundary ∂D. For a given finite configuration $(l)=\{l_1,\ldots,l_n\}$ of straight lines intersecting D denote by $\mathcal{C}_D(l)$ the family of those polygonal contours in $\mathcal{C}_D$ which belong to $\Gamma_D(l)$. Equip the space $\mathcal{C}_D$ with the Hausdorff metric. It is well known that the Hausdorff metric space on the family of compact subsets of $D\cup\partial D$ is itself compact and hence complete and separable (see e.g. proposition 1-4-4 in Matheron, 1975). The subspace $\mathcal{C}_D$ is also a metric space and inherits separability. It is easily seen not to be complete, yet it is a subspace of the compact space of compact subsets of $D\cup\partial D$ considered above and hence any point process on $\mathcal{C}_D$ is well defined (cf. Daley & Vere-Jones, 1988).

Our next step is to show that the distribution of $\mathcal{A}_D^{[\beta]}$ is of a similar form to (4), but with $\mathcal{P}$ replaced by a Poisson process on $\mathcal{C}_D$ with a suitably chosen intensity measure. To do so, define the so-called *free contour measure* Θ_D on $\mathcal{C}_D$ by setting for $C \subseteq \mathcal{C}_D$ measurable with respect to the Borel σ-field generated by the Hausdorff distance topology,

$$\Theta_D(C) = \int_{\mathrm{Fin}(L[D])} \sum_{\theta \in C \cap \mathcal{C}_D(l)} \exp(-2\ell(\theta))\, \mathrm{d}\mu^*((l)) \tag{5}$$

with Fin($L[D]$) standing for the family of finite line configurations intersecting D and where μ^* is the measure on Fin($L[D]$) given by

$$\mathrm{d}\mu^*(l_1, \ldots, l_n) := \prod_{i=1}^{n} \mathrm{d}\mu(l_i)$$

with μ defined in the discussion preceding (1). For $\beta \in \mathbb{R}$, define the exponential modification $\Theta_D^{[\beta]}$ of the free contour measure Θ_D by

$$\frac{\mathrm{d}\Theta_D^{[\beta]}}{\mathrm{d}\Theta_D}[\theta] := \exp(-\beta\ell(\theta)) \tag{6}$$

and let $\mathcal{P}_{\Theta_D^{[\beta]}}$ be the Poisson process on $\mathcal{C}_D$ with intensity measure $\Theta_D^{[\beta]}$. Then, by (5) and (1), the polygonal field $\mathcal{A}_D^{[\beta]}$ coincides in distribution with the union of contours in $\mathcal{P}_{\Theta_D^{[\beta]}}$ conditioned on the event that they are disjoint, i.e.

$$\mathcal{L}\left(\mathcal{A}_D^{[\beta]}\right) = \mathcal{L}\left(\mathcal{P}_{\Theta_D^{[\beta]}} \,\middle|\, \forall \theta, \theta' \in \mathcal{P}_{\Theta_D^{[\beta]}} : \theta \neq \theta' \Rightarrow \theta \cap \theta' = \emptyset\right) \tag{7}$$

(see section 2.2 in Schreiber, 2005). Note that the conditioning in (7) makes sense because $\Theta_D^{[\beta]}(\mathcal{C}_D)$ is finite as shown in section 2.2 in Schreiber (2005). Furthermore, for all bounded open sets D with piecewise smooth boundary, the free contour measures Θ_D as defined in (5) arise as the respective restrictions to $\mathcal{C}_D$ of the same measure Θ on $\mathcal{C} := \bigcup_{n=1}^{\infty} \mathcal{C}_{(-n,n)^2}$, in the sequel referred to as the *infinite volume free contour measure*. In the same way, we construct the infinite volume Gibbs-modified measures $\Theta^{[\beta]}$.

To place the polymer representation in the marked point process setting of section 2.2, identify a given contour collection $\{\theta_1, \ldots, \theta_k\}$ arising as a realization of $\mathcal{A}_D^{[\beta]}$, with the collection of points $x_i := \imath[\theta_i], i = 1, \ldots, k$, carrying the respective contours as their marks, where $\imath[\cdot]$ is a mapping from $\mathcal{C}_D$ to D. Even though a number of different natural candidates for $\imath[\cdot]$ could be considered, to be specific in the sequel we shall always take $\imath[\theta]$ to be the extreme left point of the contour θ, minimizing the first coordinate, with possible ties broken in an arbitrary measurable way. For formal convenience we regard the marks θ_i attached to the points $x_i \in D$ as elements of the common space $\mathcal{C}_0 := \{\theta \in \mathcal{C} \mid \imath[\theta] = 0\}$ shifted to corresponding x_i. Below, for a point $x \in D$ carrying a mark $\theta \in \mathcal{C}_0$, we shall reserve the name of *shifted mark* for the translate of the contour θ by the vector x. It is also convenient for our further purposes to admit in $\mathcal{C}_0$ the empty contour $\emptyset$. In this way, the object process on $\mathcal{C}_D$ described above is re-formulated as a marked point process on D with marks in $\mathcal{C}_0$.

It remains to endow the mark space $M = \mathcal{C}_0$ with a probability measure ν_M so that by conditioning a homogeneous Poisson process on D randomly marked with independent ν_M-distributed contours on the event of no intersection between the shifted marks, the length-interacting Arak process is obtained. For $\beta \geq 2$, this can be carried out by the random walk representation of Schreiber (2006) as follows. Let $\Theta_*^{[\beta]}$ be determined by the following construction of a $\mathcal{C}_0$-valued $\Theta_*^{[\beta]}$-distributed random element θ:

1. Simulate a continuous-time random walk Z_t governed by the following dynamics:

(i) set $Z_0 := 0$ and choose the initial direction uniformly in $(0, 2\pi)$;

(ii) between direction update events specified below move in a constant direction with speed 1;

(iii) with intensity 4 update the movement direction, choosing the angle $\phi \in (0, 2\pi)$ between the old and new directions according to the density $|\sin(\phi)|/4$.

2. Consider a killed modification $\tilde{Z}_t^{[\beta-2]}$ of Z_t by killing Z_t

(i) with constant rate $\beta - 2$;
(ii) whenever it hits its past trajectory.

3. Draw an infinite *loop-closing* half-line l^* beginning at 0 and forming with the initial segment of $(Z_t)_{t\geq 0}$ an angle $\phi^* \in (0, 2\pi)$ distributed according to the density $|\sin \phi^*|/4$.

4. If the random walk $\tilde{Z}_t^{[\beta-2]}$ hits the loop-closing half-line l^* before being killed, and the self-avoiding contour $\theta_* := \theta_*[\tilde{Z}^{[\beta-2]}; l^*]$ created by l^* and the trajectory of $\tilde{Z}_t^{[\beta-2]}$ up to the moment of hitting l^* satisfies $\imath[\theta_*] = 0$, then

(i) with probability $\exp(-[\beta+2]\ell(e^*))$ output $\theta := \theta_*$, where e^* stands for the segment of the loop-closing half-line l^* from 0 to its intersection point with $\tilde{Z}_t^{[\beta-2]}$;
(ii) otherwise output $\theta := \emptyset$.

In all remaining cases set $\theta := \emptyset$.

The algorithmic definition has the advantage of being easy to implement. The following lemma, close in spirit to lemma 5.1 in Schreiber (2006), is the main result of this section, and states the validity of the random walk construction.

Lemma 1
For $\beta \geq 2$ the polygonal Markov field $\mathcal{A}_D^{[\beta]}$ coincides in law with the union of contours carried as shifted marks by the $\mathcal{C}_0$-marked point process $\mathcal{Y}^{[\beta]}$ in D, determined by Papangelou conditional intensity

$$\lambda\left((x,\theta);\{(x_i,\theta_i)\}_{i=1}^k\right) := \begin{cases} 4\pi, & \text{if } x+\theta \cap [\bigcup_{i=1}^k x_i + \theta_i] = \emptyset, \quad x+\theta \subseteq D, \\ 0, & \text{otherwise.} \end{cases} \tag{8}$$

with respect to the product of Lebesgue measure on D and $\Theta_*^{[\beta]}$ on $\mathcal{C}_0$.

Proof. The directed nature of the random walk trajectories as constructed above requires considering for each contour θ two oriented instances $\theta^{\rightarrow}$ (clockwise) and $\theta^{\leftarrow}$ (anticlockwise). In view of the polymer representation (7) and taking into account that $\Theta_D^{[\beta]}$ arises as the restriction of $\Theta^{[\beta]}$ to $\mathcal{C}_D$, by the construction of $\Theta_*^{[\beta]}$ the assertion of the lemma will follow as soon as we show that for each $x \in \mathbb{R}^2$ and $\theta \in \mathcal{C}$ we have

$$8\pi\, \mathrm{d}x\; \mathrm{e}^{-[\beta+2]\ell(e^*)}\mathbb{P}\left(\tilde{Z}_t^{[\beta-2]} \text{ reaches } l^* \text{ and } x + \theta_*[\tilde{Z}^{[\beta-2]}; l^*] \in \mathrm{d}\theta^{\rightarrow}\right) = \Theta^{[\beta]}(\mathrm{d}\theta), \tag{9}$$

where e^* stands for the last segment of $\theta^{\rightarrow}$ counting from x as the initial vertex, which is to coincide with the x-shifted segment of the loop-closing line l^* joining its intersection point with $\tilde{Z}_t^{[\beta-2]}$ to 0; whereas $\theta_*[\tilde{Z}^{[\beta-2]}; l^*]$ is the self-avoiding contour created by l^* and the trajectory of $\tilde{Z}_t^{[\beta-2]}$ up to the moment of hitting l^*, as denoted in the construction of $\Theta_*^{[\beta]}$ above. Indeed, the same relation holds then for $\theta^{\leftarrow}$, whence adding versions of (9) for $\theta^{\rightarrow}$ and $\theta^{\leftarrow}$, which amounts to taking into account two possible directions in which the random walk can move along θ, will yield $2\Theta^{[\beta]}(\mathrm{d}\theta)$ on the right-hand side (RHS), whence (8) will follow. Observe that $\theta_*[\tilde{Z}^{[\beta-2]}; l^*]$ is shifted by x in (9) above – this is because the measure $\Theta^{[\beta]}(\cdot)$ is defined on the space of bounded polygonal contours $\mathcal{C}$ and this is where θ in the RHS of (9) belongs to whereas the measure $\Theta_*^{[\beta]}(\cdot)$, determined by the law of $\theta_*[\tilde{Z}^{[\beta-2]}; l^*]$ as showing up in the left-hand side of (9), is defined on the space $\mathcal{C}_0$ consisting of contours $\zeta \in \mathcal{C}$ with $\imath[\zeta] = 0$.

To establish (9), we observe that the probability element

$$\mathbb{P}\left(\tilde{Z}_t^{[\beta-2]} \text{ reaches } l^* \text{ and } x+\theta_*[\tilde{Z}^{[\beta-2]}; l^*] \in \mathrm{d}\theta^{\rightarrow}\right)$$

is exactly

$$\frac{1}{4[\mu\times\mu](\{(l,l^*)\,|\,l\cap l^*\in \mathrm{d}x\})}\exp(-[(\beta-2)+4]\ell(\theta\backslash e^*))\prod_{i=1}^{k}\mathrm{d}\mu(l[e_k]), \tag{10}$$

where $e_1,\ldots,e_k$ are all segments of θ including e^*, while $l[e_i]$ stands for the straight line determined by e_i. Indeed,

1. The pre-factor $[4[\mu\times\mu](\{(l,l^*)\,|\,l\cap l^*\in \mathrm{d}x\})]^{-1}$ comes from the choice of the lines containing, respectively, the initial segment of $\theta^{\rightarrow}$ (counting from x) and l^* as well as from the choice between two equiprobable directions on each of these lines.
2. For the remaining segments we use the fact that, for any given straight line l_0, $\mu(\{l\,|\,l\cap l_0\in \mathrm{d}\ell, \angle(l,l_0)\in \mathrm{d}\phi\})=|\sin\phi|\,\mathrm{d}\ell\;\mathrm{d}\phi$ with $\mathrm{d}\ell$ standing for the length element on l_0 and with $\angle(l_0,l)$ denoting the angle between l and l_0 (see proposition 3.1, as well as the argument justifying the dynamic representation of the Arak process in section 4 of Arak & Surgailis, 1989 and the proof of lemma 1 in Schreiber, 2005). Note that the direction update intensity was set to 4 to coincide with $\int_0^{2\pi}|\sin\phi|\,\mathrm{d}\phi=4$.
3. The additional prefactor $\exp(-[(\beta-2)+4]\ell(\theta\backslash e^*))$ shows up in (10) because:
   (i) the polygonal path $\theta\backslash e^*$ arises as the x-shifted trajectory of the random walk $\tilde{Z}^{[\beta-2]}$ whose survival probability along $\theta\backslash e^*$ is precisely $\exp(-[(\beta-2)+4]\ell(\theta\backslash e^*))$;
   (ii) the segment e^* of the x-shifted loop-closing line l^* is not generated by $\tilde{Z}^{[\beta-2]}$ and hence is not subject to killing.

Recalling that

$$\Theta(\mathrm{d}\theta)=\exp(-2\ell(\theta))\prod_{i=1}^{k}\mathrm{d}\mu(l[e_k])$$

[see (5)] and observing that $[\mu\times\mu](\{(l,l^*)\,|\,l\cap l^*\in \mathrm{d}x\})=2\pi\,\mathrm{d}x$ as follows by standard integral geometry (cf. proposition 3.1 in Arak & Surgailis, 1989), we see that the expression in (10) coincides with $(1/8\pi\,\mathrm{d}x)\exp([\beta+2]\ell(e^*))\exp(-\beta\ell(\theta))\Theta(\mathrm{d}\theta)$, which yields the required relation (9) upon recalling the definition of $\Theta^{[\beta]}$ [see (6)]. The proof is complete.

4. Perfect simulation using spatial birth-and-death processes

In this section, we consider two exact sampling methods based on the classic idea to simulate from planar point process models by means of running a spatial birth-and-death process. Coupling from the past techniques (CFTP) for point processes were introduced by Kendall (1998) for a special model. The generalization to locally stable point processes can be found in Kendall & Møller (2000), and to marked patterns in Van Lieshout & Stoica (2006). In the context of spatial interpolation and cluster modelling, Van Lieshout & Baddeley (2002) present an adaptive CFTP algorithm, whereas Lund & Thönnes (2004) use auxiliary marks in a similar framework.

Algorithm 1 (CFTP).

1. *Initialize $T=1$, and let $\mathcal{Y}(0)$ be a realization of a Poisson process of rate 4π in D, marked independently and indentically distributed (i.i.d) according to $\nu_M=\Theta_*^{[\beta]}$.*

2. *Extend* $\mathcal{Y}(\cdot)$ *backwards to time* $-T$ *by means of a spatial birth-and-death process with birth rate* $4\pi\,\mathrm{d}x\Theta_*^{[\beta]}(\mathrm{d}\theta)$ *and unit death rate.*
3. *Generate* $L^{-T}(\cdot)$ *(lower process) and* $U^{-T}(\cdot)$ *(upper process) forwards in time as follows:*
   - *(i) set* $L^{-T}(-T)=\emptyset$ *and* $U^{-T}(-T)=\mathcal{Y}(-T)$*;*
   - *(ii) if* $\mathcal{Y}(\cdot)$ *experiences a backward birth at time t, i.e.* $\mathcal{Y}(t-)=\mathcal{Y}(t)\cup\{(x,\theta)\}$ *for some* $(x,\theta)\notin\mathcal{Y}(t)$*, where* $\mathcal{Y}(t-)$ *denotes the state just prior to t, delete* (x,θ) *from* $L^{-T}(t-)$ *and* $U^{-T}(t-)$*;*
   - *(iii) if* $\mathcal{Y}(\cdot)$ *experiences a backward death at time t, i.e.* $\mathcal{Y}(t-)=\mathcal{Y}(t)\setminus\{(x,\theta)\}$ *for some* $(x,\theta)\in\mathcal{Y}(t)$*, the marked point* (x,θ) *is added to* $L^{-T}(t-)$ *iff*

$$[x+\theta]\cap\bigcup_{(x_i,\,\theta_i)\in U^{-T}(t-)}[x_i+\theta_i]=\emptyset,\quad [x+\theta]\subseteq D$$

and to $U^{-T}(t-)$ *iff*

$$[x+\theta]\cap\bigcup_{(x_i,\,\theta_i)\in L^{-T}(t-)}[x_i+\theta_i]=\emptyset,\quad [x+\theta]\subseteq D.$$

4. *If* $U^{-T}(0)=L^{-T}(0)$ *stop. Else set* $T=2T$ *and go to 2.*
5. *Return* $U^{-T}(0)$.

The clan of ancestors algorithm of Fernández *et al.* (2002) is similar in flavour. It has the advantage of avoiding the birth of marked points that will have no influence on the final outcome, but does not exploit the repulsive behaviour of the hard core contour process. It tends to be better than coupling from the past for low intensities, worse for higher ones (cf. Van Lieshout & Stoica, 2006).

Algorithm 2 (clan of ancestors).

1. *Let* $\mathcal{Y}(0)$ *be a realization of a Poisson process of rate* 4π *in D, marked i.i.d. according to* $\nu_M=\Theta_*^{[\beta]}$*. Initialize the clan of ancestors as* $A=\mathcal{Y}(0)$*.*
2. *Extend* $\mathcal{Y}(\cdot)$ *backwards by means of a spatial birth-and-death process with birth rate* $4\pi\,\mathrm{d}x\Theta_*^{[\beta]}(\mathrm{d}\theta)$ *and unit death rate. At each death incident* $\mathcal{Y}(t-)=\mathcal{Y}(t)\setminus\{(x,\theta)\}$ *for some* $t<0$ *and* $(x,\theta)\in A\cap\mathcal{Y}(t)$*, add the marked points* $(x',\theta')\in\mathcal{Y}(t-)$ *for which* $x'+\theta'\cap x+\theta\neq\emptyset$ *to A. The backwards sweep ends when* $A_t=A\cap\mathcal{Y}(t)=\emptyset$*. The stopping time thus obtained is denoted by* $-T$*.*
3. *Generate* $\mathcal{Z}(\cdot)$ *forwards in time as follows:*
   - *(i) set* $\mathcal{Z}(-T)=\emptyset$*;*
   - *(ii) if* $\mathcal{Y}(\cdot)$ *experiences a backward birth at time t, i.e.* $\mathcal{Y}(t-)=\mathcal{Y}(t)\cup\{(x,\theta)\}$ *for some* $(x,\theta)\notin\mathcal{Y}(t)$*, delete* (x,θ) *from* $\mathcal{Z}(t-)$*;*
   - *(iii) if* $\mathcal{Y}(\cdot)$ *experiences a backward death at time t, i.e.* $\mathcal{Y}(t-)=\mathcal{Y}(t)\setminus\{(x,\theta)\}$ *for some* $(x,\theta)\in A_t$*, the marked point* (x,θ) *is added to* $\mathcal{Z}(t-)$ *iff*

$$[x+\theta]\cap\bigcup_{(x_i,\,\theta_i)\in\mathcal{Z}(t-)}[x_i+\theta_i]=\emptyset,\quad [x+\theta]\subseteq D;$$

if $(x,\theta)\notin A_t$ *then* $\mathcal{Z}(t)=\mathcal{Z}(t-)$ *remains unchanged.*

4. *Return* $\mathcal{Z}(0)$.

Some realizations obtained by algorithms 1 and 2 implemented in C++ using the library MPPLIB (Steenbeek *et al.*, 2002–2003) are presented in Figs 1 and 2. The execution time is in the order of seconds for Fig. 1, minutes for those in Fig. 2 on a 2.1 GHz desktop computer.

Both algorithms 1 and 2 can be placed in the general framework discussed in (Van Lieshout & Stoica, 2006) as soon as we are able to verify the conditions **(M1)** and **(M2)** for the Papangelou conditional intensity $\lambda(\cdot;\cdot)$ as given in (8), as well as the repulsivity of $\lambda(\cdot;\cdot)$ needed for the validity of algorithm 1. All these required relations are, however, self-evident, which leads us to the following lemma which concludes the current section.

Lemma 2
The polygonal Markov field $\mathcal{A}_D^{[\beta]}$ coincides in distribution with both $U^{-T}(0)=L^{-T}(0)$ as constructed in algorithm 1 and with the output $\mathcal{Z}(0)$ of algorithm 2.

5. Conclusion

In this paper, we designed perfect simulation algorithms for length-interacting Arak polygonal Markov fields observed in bounded planar windows. To do so, we re-formulated the model as a marked point process of hard objects, derived the mark distribution, and specialized the coupling from the past and clan of ancestors algorithms developed for (marked) point processes to our particular model. The clan of ancestors method is the fastest for low intensities; coupling from the past applies to a wider range of parameter values (β and window size). We aim at constructing high-temperature perfect samplers as well, which is the subject of our current work in progress.

An alternative to spatial birth-and-death-based simulation is to use the Metropolis–Hastings framework (Clifford & Nicholls, 1994). In contrast to a spatial birth-and-death sampler which accepts all proposed transitions, a Metropolis–Hastings algorithm accepts a new state with a probability that depends on the likelihood ratio of the new state compared with the current one. Note that transitions do not need to be limited to births and deaths; for example, one might wish to alter the angle between two edges or the position of a vertex. Although such flexibility may be very desirable in practice to improve the mixing time, it also implies that it may be harder to design a perfect version than for the dynamics discusssed in section 4. Even in the simplest case with birth-and-death proposals only, one has to discretize D in order to ensure that deaths are accepted with probability 1, and be careful which marked point to delete, so as to maintain the set inclusion order between the upper and lower processes U and L (cf. algorithm 1); furthermore, simulation studies by Van Lieshout & Stoica (2006) suggest the increased complexity is not repaid by increased efficiency, so we do not pursue the topic here. For further details on perfect Metropolis–Hastings sampling, see Kendall & Møller (2000) or Van Lieshout & Stoica (2006).

Acknowledgements

We gratefully acknowledge support from the EC 6th Framework Programme Priority 2 Information Society Technology Network of Excellence MUSCLE (Multimedia Understanding through Semantics, Computation and Learning). Van Lieshout gratefully acknowledges financial support from Eurandom, Eindhoven, the Netherlands. Schreiber also wishes to express his gratitude for support from the Polish Minister of Scientific Research and Information Technology grant and to warmly thank for hospitality of the Centrum voor Wiskunde en Informatica (CWI), Amsterdam, the Netherlands, where this research was carried out. He also addresses his special thanks to the RDSES programme of the European Science Foundation, which provided numerous sources of inspiration for initiating this research.

References

Arak, T. (1982). On Markovian random fields with finite number of values. *4th USSR-Japan symposium on probability theory and mathematical statistics, Abstracts of Communications*, Tbilisi.

Arak, T. & Surgailis, D. (1989). Markov fields with polygonal realisations. *Probab. Theory Related Fields* **80**, 543–579.

Arak, T. & Surgailis, D. (1991). Consistent polygonal fields. *Probab. Theory Related Fields* **89**, 319–346.

Arak, T., Clifford, P. & Surgailis, D. (1993). Point-based polygonal models for random graphs. *Adv. Appl. Probab.* **25**, 348–372.

Clifford, P. & Middleton, R. D. (1989). Reconstruction of polygonal images. *J. Appl. Statist* **16**, 409–422.

Clifford, P. & Nicholls, G. (1994). A Metropolis sampler for polygonal image reconstruction. Available at: http://www.stats.ox.ac.uk/~clifford/papers/met_poly.html (accessed December 22, 1994).

Daley, D. J. & Vere-Jones, D. (1988). *An introduction to the theory of point processes*. Springer Series in Statistics, Springer-Verlag, New York, NY.

Fernández, R., Ferrari, P. & Garcia, N. (2002). Perfect simulation for interacting point processes, loss networks and Ising models. *Stochastic Process Appl.* **102**, 63–88.

Kendall, W. S. (1998). Perfect simulation for the area-interaction point process. In *Probability towards 2000* (eds. L. Accardi & C. Heyde, 218–234, Springer, New York, NY.

Kendall, W. S. & Møller, J. (2000). Perfect simulation using dominating processes on ordered spaces, with application to locally stable point processes. *Adv. Appl. Probab.* **32**, 844–865.

Kluszczyński, R., Van Lieshout, M. N. M. & Schreiber, T. (2005). An algorithm for binary image segmentation using polygonal Markov fields. In *Image Analysis and Processing, Proceedings of the 13th International Conference on Image Analysis and Processing* (eds F. Roli & S. Vitulano), *Lecture Notes in Comput. Sci.* **3615**, 383–390.

Kluszczyński, R., Van Lieshout, M. N. M. & Schreiber, T. (2006). Image segmentation by polygonal Markov fields. *Ann. Inst. Statist. Math.* (in press).

Van Lieshout, M. N. M. (2000). *Markov point processes and their applications*. Imperial College Press, London.

Van Lieshout, M. N. M. & Baddeley, A. J. (2002). Extrapolating and interpolating spatial patterns. In *Spatial cluster modelling* (eds A. B. Lawson & D. G. T. Denison). Chapman and Hall/CRC, Boca Raton. FL 61–86.

Van Lieshout, M. N. M. & Stoica, R. S. (2006). Perfect simulation for marked point processes. *Comput. Statist. Data Anal.* **51**, 679–698.

Lund, J. & Thönnes, E. (2004). Perfect simulation and inference for point processes given noisy observations. *Comput. Statist.* **19**, 317–336.

Matheron, G. (1975). *Random sets and integral geometry*. Wiley, New York, NY.

Nicholls, G. K. (2001). Spontaneous magnetisation in the plane. *J. Statist. Phys.* **102**, 1229–1251.

Paskin, M. A. & Thrun, S. (2005). Robotic mapping with polygonal random fields. In *Proceedings of the 21st Annual Conference on Uncertainty in Artificial Intelligence (UAI-05)*, 450–458. AUAI Press, Arlington, VA.

Propp, J. G. & Wilson, D. B. (1996). Exact sampling with coupled Markov chains and applications to statistical mechanics. *Random Struct. Algorithms* **9**, 223–252.

Ripely, B. D. & Kelly, F. P. (1977). Markov point processes. *J. London Math. Soc.* **15**, 188–192.

Schreiber, T. (2005). Random dynamics and thermodynamic limits for polygonal Markov fields in the plane. *Adv. Appl. Probab.* **37**, 884–907.

Schreiber, T. (2006). Dobrushin–Kotecký–Shlosman theorem for polygonal Markov fields in the plane. *J. Statist. Phys.* **123**, 631–684.

Steenbeek, A. G., Van Lieshout, M. N. M. & Stoica, R. S. with contributions from Gregori, P. & Berthelsen, K. K. (2002–2003). MPPBLIB, a C++ library for marked point processes. CWI, Amsterdam.

Received February 2006, in final form October 2006

T. Schreiber, Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, ul. Chopina 12–18, 87–100 Toruń, Poland.
E-mail: tomeks@mat.uni.torun.pl